NOTES ON HARMONIC ANALYSIS PART I: THE FOURIER TRANSFORM

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Abstract. Fourier Transforms is a first in a series of monographs we present on harmonic analysis. Harmonic analysis is one of the most fascinating areas of research in mathematics. Its centrality in the development of many areas of mathematics such as partial di□erential equations and integration theory and its many and diverse applications in sciences and engineering fields makes it an attractive field of study and research.

The purpose of these notes is to introduce the basic ideas and theorems of the subject to students of mathematics, physics or engineering sciences. Our goal is to illustrate the topics with utmost clarity and accuracy, readily understandable by the students or interested readers. Rather than providing just the outlines or sketches of the proofs, we have actually provided the complete proofs of all theorems. This will illuminate the necessary steps taken and the machinery used to complete each proof.

The prerequisite for understanding the topics presented is the knowledge of Lebesgue measure and integral. This will provide ample mathematical background for an advanced undergraduate or a graduate student in mathematics.

1. Fourier Transforms for $L^1(\mathbb{R})$

Definition 1.1. For $f \in L^1(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}dx \tag{1.1}$$

for all real $y \in \mathbb{R}$

It is easy to see that Fourier transform is a lineaer operator, i.e., $(f + g)\hat{}(y) = \hat{f}(y) + \hat{g}(y)$ and $(f)\hat{}(y) = \hat{f}(y)$ Also using simple integration tech0R3010.9091Tf -298.08-6R3010.9091Tf 701(4-5.39795

Theorem 1.2 (Riemann-Lebesgue Lemma). If $f \in L^1(\mathbb{R})$, then $\hat{f}(y) \to 0$ as $y \to \pm\infty$.

Proof: First suppose that f is a characteristic function of an interval $[a, b]$. Its Fourier transform is

$$\int_a^b e^{-ixy}dx = \frac{e^{-iay} - e^{-iby}}{iy}, \quad y \neq 0,$$

which tends to zero. Therefore, a linear combination of characteristic functions of intervals, i.e., a step function, satisfies the Riemann-Lebesgue lemma. Such functions are also dense in $L^1(\mathbb{R})$. Now let $f \in L^1(\mathbb{R})$ and let $f_n \in L^1(\mathbb{R})$ be a sequence of step functions such that $f_n \to f$ in $L^1(\mathbb{R})$. Then

$$|\hat{f_n}(y) - \hat{f}(y)| = |(f_n - f)\hat{\ }(y)| \leq ||f_n - f||_1 \to 0.$$

Note that the limit is uniform in $y \in \mathbb{R}$. Since

$$|\hat{f}(y)| \leq |\hat{f_n}(y) - \hat{f}(y)| + |\hat{f_n}(y)|,$$

we can choose n large enough so that the first term on the right is small and then for that fixed n, we let $|y|$ large enough so that the second term is also small. This completes the proof. □

Theorem 1.3. Suppose that $f(x)(1 + |x|)$ is integrable. Then,

$$(\hat{f})'(y) = (-ixf(x))\hat{\ }(y). \tag{1.2}$$

Proof: Note that, by assumption, both f and $xf(x)$ are integrable. We write

$$\begin{aligned}(\hat{f})'(y) &= \lim_{h\to 0}\int_{-\infty}^{\infty} f(x)\frac{e^{-ix(y+h)} - e^{-ixy}}{h}dx \\ &= \lim_{h\to 0}\int_{-\infty}^{\infty} f(x)e^{-ixy}\frac{e^{-ixh} - 1}{h}dx\end{aligned}$$

Note that the integrand converges to $f(x)e^{-ixy}(-ix)$ pointwise as $h \to 0$ and $|f(x)e^{-ixy}\frac{e^{-ixh}-1}{h}| \leq |xf(x)|$ for all small $|h|$.[1] Hence, by Lebesgue's dominated convergence theorem ,

$$(\hat{f})'(y) = \lim_{h\to 0}\int_{-\infty}^{\infty} f(x)e^{-ixy}\frac{e^{-ixh} - 1}{h}dx = \int_{-\infty}^{\infty}(-ixf(x))e^{-ixy}dx \quad \square$$

Theorem 1.4. If f is continuously differentiable with compact support, then

$$(f')\hat{\ }(y) = iy\hat{f}(y). \tag{1.3}$$

Proof: Integration by parts. □

Definition 1.2. The convolution of f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt, \tag{1.4}$$

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[1] Estimating the remainder (both Lagrange form and integral form) of Taylor's series for e^{ix} we obtain the estimation

$$|e^{ix} - \sum_{k=0}^{n}\frac{(ix)^k}{k!}| \leq \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right).$$

Note that the first estimate is better for small $|x|$, while the second is better for large $|x|$. Choosing $n = 0$ and considering small $|h|$ we get the inequality in context.

whenever the integral exists.

In the following, $C(\mathbb{R})$ denotes the space of all continuous functions on $\mathbb{R}$ with $||f||_C =$ $sup_{x\in\mathbb{R}}|f(x)| \quad \infty$ and $C_0(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ that vanishes at infinity, i.e., for any . 0, there is a compact $F \subset \mathbb{R}$ such that $|f(x)|$. for $x \notin F$ Then by F. Riesz' theorem, $(C_0(\mathbb{R}))^* = M(\mathbb{R})$, where $M(\mathbb{R})$

Also note that the expression on the right belongs to $L^1(\mathbb{R})$. Hence, the integral

$$\int\left(\int |f(x-t)g(t)|dx\right)dt = ||f||_1||g||_1$$

exists as a finite number. Therefore, by Fubini's theorem the integral

$$\int\left(\int |f(x-t)g(t)|dt\right)dx$$

exists and is equal to $||f||_1||g||_1$. This implies that $\int |f(x-t)g(t)|dt$ exists a.e. and belongs to L^1.

To prove $(f*g)\hat{}(y) = \hat{f}(y)\cdot\hat{g}(y)$, we observe that

$$\begin{aligned}(f*g)\hat{}(y) &= \int\left(\int f(x-t)g(t)dt\right)e^{-ixy}dx \\ &= \int g(t)e^{-ity}\left(\int f(x-t)e^{-iy(x-t)}dx\right)dt = \hat{f}(y)\hat{g}(y)\end{aligned}$$

The change in the order of integration is justified by Fubini's theorem. □

It is easy to see that convolution obeys the commutative and distributive laws of algebra in $L^1(\mathbb{R})$, i.e., $f*g = g*f$ and $f*(g+h) = f*g + f*h$. The natural question is whether there is a multiplicative identity, i.e., given $f \in L^1(\mathbb{R})$, is there $e \in L^1(\mathbb{R})$ such that $f*e = f$? The answer is, in general, no since convolution exhibits continuity property and cannot be equal to a discontinuous f. However, we may seek a sequence of functions e_n, called approximate identity, with the property that $e_n * f \to f$.

Definition 1.3. An approximate identity e_n on $\mathbb{R}$ is a sequence of functions e_n such that $e_n \geq 0$, $\int e_n(x)dx = 1$, and for each $\delta > 0$,

$$\lim_{n\to\infty}\int_{|x|>\delta} e_n(x)dx = 0$$

Theorem 1.7. If $f \in C_0(\mathbb{R})$, then $e_n * f \to f$ uniformly. If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then $e_n * f \to f$ in $L^p(\mathbb{R})$. If $f \in L^\infty(\mathbb{R})$, then $e_n * f \to f$ in the weak* topology of $L^\infty(\mathbb{R})$ as a dual of $L^1(\mathbb{R})$, that is, $\int (e_n * f)(x)g(x)dx \to \int f(x)g(x)dx$ for all $g \in L^1(\mathbb{R})$.

Proof: Note that if $f \in C_0(\mathbb{R})$, then f is uniformly continuous on $\mathbb{R}$ and for any given $\epsilon > 0$, there is a $\delta > 0$ such that for any t with $|t| < \delta$, $|f(x-t) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}|(e_n * f)(x) - f(x)| &\leq \int_{\mathbb{R}} |f(x-t) - f(x)|e_n(t)dt \\ &= \int_{|t|<\delta} |f(x-t)-f(x)|e_n(t)dt + \int_{|t|\geq\delta}|f(x-t)-f(x)|e_n(t)dt \\ &\leq \epsilon + 2M\int_{|t|>\delta} e_n(t)dt,\end{aligned}$$

where $M = \sup_{x\in\mathbb{R}} |f(x)|$. Since $\lim_{n\to\infty} \int_{|x|>\delta} e_n(x)dx = 0$, $e_n * f \to f$ uniformly. In the case of $f \in L^\infty(\mathbb{R})$, the proof is similar.

If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then

$$\begin{aligned}
\int_{\mathbb{R}} |(e_n * f)(x) - f(x)|^p dx &\le \int_{\mathbb{R}} \left|\int_{\mathbb{R}} (f(x-t) - f(x))e_n(t)dt\right|^p dx \\
&\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-t) - f(x)|^p |e_n(t)|dx\right)dt \\
&= \int_{\mathbb{R}} ||f(\cdot - t) - f(\cdot)||_p^p |e_n(t)|dt
\end{aligned}$$

Given any $\epsilon > 0$, there is a $\delta > 0$ such that $||f(\cdot - t) - f(\cdot)||_p < \epsilon$ whenever $|t| < \delta$. Hence,

$$\begin{aligned}
&\int_{R} ||f(\cdot - t) - f(\cdot)||_p^p |e_n(t)|dt \\
= &\int_{|t|<\delta} ||f(\cdot - t) - f(\cdot)||_p^p e_n(t)dt + \int_{|t|\ge\delta} ||f(\cdot - t) - f(\cdot)||_p^p e_n(t)dt \\
\le &\ \epsilon^p + 2||f||_p^p \int_{|t|>\delta} e_n(t)dt
\end{aligned}$$

Since $\lim_{n\to\infty} \int_{|x|>\delta} e_n(x)dx = 0$, the result follows. □

Theorem 1.8. If f has compact support and a continuous derivative, and $g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ has a continuous derivative.

Proof: First, we prove

$$\frac{d}{dx}\left(\int f(x-t)g(t)dt\right) = \int f'(x-t)g(t)dt,$$

which is showing that

$$\lim_{h\to 0} \int_{-\infty}^{\infty} \left(\frac{f(x+h-t) - f(x-t)}{h}\right)g(t)dt = \int f'(x-t)g(t)dt$$

Note that the integrand on the left converges to $f'(x-t)g(t)$ pointwise (in t) as $h \to 0$. Moreover, $\frac{f(x+h-t)-f(x-t)}{h} = f'(c)$, where c is between $x+h-t$ and $x-t$. If f has compact support S, then so does f'. Therefore, $|\frac{f(x+h-t)-f(x-t)}{h}| = |f'(c)| \le sup_{c\in S}|f'(c)| \le M$ with some $M > 0$ for all $t \in (-\infty, \infty)$. Now the desired limit follows from Lebesgue's dominated convergence theorem.

To prove that $\int f'(x-t)g(t)dt$ is continuous, we note that

$$\begin{aligned}
&\left|\int f'(x+h-t)g(t)dt - \int f'(x-t)g(t)\right| \\
= &\left|\int f'(t)(g(x+h-t) - g(x-t))dt\right| \\
\le &\ ||f'||_c ||g(\cdot + h) - g(\cdot)||_1
\end{aligned}$$

Then the (uniform) continuity of $\int f'(x-t)g(t)dt$ follows from the continuity of g in mean.

□

The following corollary follows immediately from Theorems 1.7 and 1.8.

Corollary 1.1. **Let** $e_{\mathbf{n}}$

That is, f is the inverse Fourier transform of $\hat{f}$

Proof: If $t \neq 0$, let $(x) = f(x + t)$ If $f(x)$ satisfies $|f(x) - f(t)| \leq K|x - t|^{\alpha}$ for x near $t \neq 0$)

Proof: Assuming that x 0 and integrating

$$I = \frac{1}{2} \int_{\Gamma_R} \frac{2}{1 + z^2} e^{ixz} dz,$$

where Γ_R consists of the upper semicircle C_R and the line segment $[-\ ,\ ]$ on the x-axis, we see that

$$I = \quad es_{z=i}$$

In particular, let $z = \frac{-iy}{2}$ Then we have

$$\int_{-\infty}^{\infty} e^{-x^2-ixy}dx = \sqrt{\ } e^{-y^2/4} \qquad \square$$

Theorem 1.13 (Inversion Theorem). Let $f \in L^1(\mathbb{R})$, and $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}(y)e^{ixy}dy \tag{1.8}$$

for almost all real $x \in \mathbb{R}$. The integral is commonly known as the inverse Fourier transform.

Proof: Consider the Gauss-Weierstrass Kernel, $(x, \ ) = \frac{1}{\sqrt{\pi\alpha}}e^{-\frac{x^2}{\alpha}}$ A straightforward calculation shows that $(\cdot, \ )\hat{}(t) = e^{-\frac{\alpha t^2}{4}}$ By integrating $\hat{f}$ against $\hat{}$, and then applying Fubini's theorem and the fact that $(x, \ )$ is an approximate identity, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{f}(\ )e^{i\xi x} \quad (\cdot, \ )\hat{}(\ )d \\
= &\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt\right)e^{i\xi x}e^{-\frac{\alpha\xi^2}{4}}d \\
= &\int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} e^{-\frac{\alpha\xi^2}{4}}e^{-i\xi(t-x)}d\ \right)dt \\
= &\int_{-\infty}^{\infty} f(t)2 \quad (t-x, \ )dt \\
= \ 2&\int_{-\infty}^{\infty} f(x-t) \quad (t, \ )dt \to 2 \ f(x) \ a\ e \quad \text{as} \quad \to 0^+
\end{aligned}$$

On the other hand, by Lebesgue's dominated convergence theorem,

$$\lim_{\alpha\to 0^+}\int_{-\infty}^{\infty} \hat{f}(\ )e^{i\xi x} \quad (\cdot, \ )\hat{}(\ )d \ = \int_{-\infty}^{\infty} \hat{f}(\ )e^{i\xi x}d$$

The theorem follows. $\square$

As an application of the inversion theorem, we now prove that the Fourier transform of a product is the convolution of the Fourier transforms.

Theorem 1.14. Assume that $f, g \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$ (or $\hat{g} \in L^1(\mathbb{R})$) Then,

$$(fg)\hat{}(x) = \frac{1}{2}(\hat{f} * \hat{g})(x) \tag{1.9}$$

Proof: By the inversion theorem, f is bounded and so, $fg \in L^1(\mathbb{R})$ Hence,

$$
\begin{aligned}
(fg)\hat{}(x) &= \int_{-\infty}^{\infty} f(y)g(y)e^{-ixy}dy \\
&= \int_{-\infty}^{\infty} g(y)e^{-ixy}(\frac{1}{2}\int_{-\infty}^{\infty}\hat{f}(t)e^{iyt}dt)dy \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\hat{f}(t)(\int_{-\infty}^{\infty} g(y)e^{-ixy}e^{iyt}dy)dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\hat{f}(t)\hat{g}(x-t)dt \\
&= \frac{1}{2}(\hat{f}*\hat{g})(x)
\end{aligned}
$$

The change in the order of integration is justified by Fubini's theorem, since due to boundedness of $\hat{f}$, $\hat{f}g \in L^1(\mathbb{R})$ □

We now investigate the question of uniqueness of Fourier transform, i.e, $\hat{f} = \hat{g}$ implies $f = g$ To show this, since Fourier transform is a linear operator, it su□ces to show that $\hat{f} = 0$ implies $f = 0$ a.e.

Theorem 1.15 (Uniqueness Theorem). If $f \in L^1(\mathbb{R})$ and $\hat{f} = 0$ everywhere ($\hat{f}$ is always continuous), then $f = 0$ a.e.

Proof: Let $e_n(x)$ be an approximate identity with compact support and continuous derivative. By Theorem 1.6, $(e_n * f)\hat{} = \hat{e_n}\hat{f} = 0$ everywhere. Since by Theorem 1.8, $e_n * f$ is continuous and di□erentiable, by the inversion theorem, $e_n * f = 0$ everywhere. But by Theorem 1.7, $e_n * f \to f$ in $L^1(\mathbb{R})$; so it follows that $f = 0$ a.e. □

Definition 1.4. For $\ \in M(\mathbb{R})$ (bounded Borel measure on $\mathbb{R}$, i.e., $|\ |(\mathbb{R})\ \ \infty$), define the Fourier-Stieltjes transform $\hat{}(y)$ as

$$\hat{}(y) = \int_{-\infty}^{\infty} e^{ixy}d\ (x)$$

Clearly, the Fourier-Stieltjes transform defines a bounded linear transform from $M(\mathbb{R})$ to $\mathbb{C}$

Theorem 1.16 (Uniqueness Theorem). If $\hat{}(y) = 0$ for a.e. y, then $\ = 0$

Proof: Since $(C_0(\mathbb{R}))^* = M(\mathbb{R})$, to prove $\ = 0$ we need only to show that for all $\ \in C_0(\mathbb{R})$, $\int\ (t)d\ (t) = 0$ This is equivalent to showing that for all $\ \in C_0(\mathbb{R})$, $(\ *\ )(0) = 0$, where $(\ *\ )(x) = \int_{-\infty}^{\infty}\ (x-t)d\ (t)$ Observe also that $\ (x) \in C_0(\mathbb{R})$ if and only if $\ (-x) \in C_0(\mathbb{R})$.

Assume that $\hat{} = 0$ Then for all $f \in L^1(\mathbb{R})$, $\hat{f} *\ (x) = (f *\ \hat{})(x) = 0$ Hence, if we prove that $\{\hat{f} : f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$, then for each $\ \in C_0(\mathbb{R})$ there is $f_n \in L^1(\mathbb{R})$ such that $\hat{f_n} \to\ $ in $C_0(\mathbb{R})$ Since $\hat{f_n} *\ (x) \to\ *\ (x)$ at each x, $\ *\ (x) = 0$

To show that $\{\hat{f} : f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$, we let

$$F(x) = \frac{1}{\sqrt{2}}\left(\frac{\sin(x\ 2)}{x\ 2}\right)^2$$

and let $F_\rho(x) = \rho F(\rho x)$ Consider the integral

$$(\ * F_\rho)(x) = \frac{2}{\rho}\int_{-\infty}^{\infty} (x-u)\frac{\sin^2(\rho u\ 2)}{u^2}du$$

Define

$$\mathcal{F} = \{(\ * F_\rho)(x) : \ \in C_0(\mathbb{R})\ \ L^1(\mathbb{R}); \rho\ \ 0\}$$

Clearly, $\mathcal{F}$ is a subset of $C_0(\mathbb{R})\ \ L^1(\mathbb{R})$ and is dense in $C_0(\mathbb{R})$

Let $\ \in C_0(\mathbb{R})\ \ L^1(\mathbb{R})$ Then $(\ * F_\rho)\hat{}(y) = \hat{}(y)(F_\rho)\hat{}(y)$ Since $\ \in L^1$, $\hat{} \in C_0(\mathbb{R})$ Moreover,

$$F_\rho\hat{}(y) = \begin{cases} 1 - \frac{|y|}{\rho} & \text{if } |y| \leq \rho \\ 0 & \text{if } |y|\ \ \rho \end{cases}$$

belongs to $L^1(\mathbb{R})$ Therefore, $(\ * F_\rho)\hat{} \in L^1(\mathbb{R})$ It follows from the inversion theorem that $\ * F_\rho$ is the Fourier transform of a function in $L^1(\mathbb{R})$ Hence, $\mathcal{F}$ is a subset of $\{\hat{f} : f \in L^1(\mathbb{R})\}$ Since $\mathcal{F}$ is dense in $C_0(\mathbb{R})$, $\{\hat{f} : f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$ □

2. Kernels on $\mathbb{R}$

We define the Dirichlet, Fejér, and Poisson kernels on $\mathbb{R}$ by defining their Fourier transforms, see H. Helson [2].

$$\hat{D}_t(y) = \begin{cases} 1 & \text{if } |y| \leq \end{cases}$$

$$
\begin{aligned}
(\hat{D}_t * \hat{D}_t)(y) &= \int_{-\infty}^{\infty} \hat{D}_t(y-\alpha)\hat{D}_t(\alpha)\,d\alpha \\
&= \int_{-t}^{t} \hat{D}_t(y-\alpha)\,d\alpha \\
&= \int_{y-t}^{y+t} \hat{D}_t(u)\,du
\end{aligned}
$$

To calculate the last integral, we consider two cases. If $|y| \geq 2t$, then the intervals $[y-t, y+t]$ and $[-t, t]$ are disjoint so that the integral equals zero; if $|y| < 2t$, then either $y+t$ or $y-t$ is in $(-t, t)$, but not both, so that the integral equals $2t-|y|$. Combining both results we get,

$(\hat{D}_t * \hat{D}_t)(y) =$

$$
\int_{y-t}^{y+t} \hat{D}_t(u)\,du = \begin{cases} 2t-|y| & \text{if } |y| \leq 2t \\ 0 & \text{if } |y| > 2t \end{cases} = 2t\hat{K}_{2t}(y)
$$

Also, by Theorem 1.14 we have that,

$$
\begin{aligned}
(D_t \cdot D_t)\hat{}(x) &= \frac{1}{2\pi}(\hat{D}_t * \hat{D}_t)(x) \\
&= \frac{1}{2\pi}(2t\hat{K}_{2t}(x))
\end{aligned}
$$

Therefore, it follows from the inversion theorem that $(D_t \cdot D_t)(x) = \frac{1}{2\pi}(2tK_{2t}(x))$, or

$$
2tK_{2t}(x) = \frac{1}{2\pi}(2\pi D_t(x))^2
$$

Hence, we obtain the Fejér kernel

$$
K_t(x) = \frac{1}{2\pi t}\left(\frac{\sin(\frac{tx}{2})}{\frac{x}{2}}\right)^2
$$

$K_t(x)$ is positive and integrable. Its Fourier transform is the function $\hat{K}_t(y)$ by the inversion theorem. Moreover, $\int K_t(x)dx = 1$ because $\hat{K}_t(y) = 1$ at $y = 0$. For any $\delta > 0$,

$$
\int_{|x|>\delta} K_t(x)dx \leq \frac{1}{2\pi t}\int_{|x|>\delta} \frac{4}{x}
$$

A direct computation of the inverse Fourier transform of $\hat{P}_u(y)$ gives

$$
\begin{aligned}
P_u(x) &= \frac{1}{2}\int_{-\infty}^{\infty} \hat{P}_u(y)e^{ixy}dy \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-u|y|}e^{ixy}dy \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{uy}e^{ixy}dy + \frac{1}{2}\int_{0}^{\infty} e^{-uy}e^{ixy}dy \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{y(u+ix)}dy + \frac{1}{2}\int_{0}^{\infty} e^{-y(u-ix)}dy \\
&= \frac{1}{2}\left(\frac{1}{u+ix} + \frac{1}{u-ix}\right) \\
&= \frac{u}{(u^2}
\end{aligned}
$$

Definition 2.1. For any $f \in L^p(\mathbb{R}), 1 \le p < \infty,$ we define the Poisson integral of f as

$$F(x+iu) = P_u * f(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{uf(s)}{u^2+(x-s)^2}ds \tag{2.2}$$

Since $P_u \in L^q(\mathbb{R}), q$ conjugate exponent of p, $F(x+iu)$ is defined as a continuous function of x.[6] Moreover, $F(x+iu)$ provides a harmonic extension of f to the upper half plane. This can be verified directly.

Theorem 2.2. The Poisson integral has a semigroup property: $P_u * P_v = P_{u+v}$ for all positive u and v.

Proof: We have that

$$\begin{aligned}(P_u \hat{*} P_v)(y) &= \hat{P_u}(y)\cdot\hat{P_v}(y) = e^{-u|y|}\cdot e^{-v|y|}\\ &= e^{-(u+v)|y|} = \hat{P_{u+v}}(y)\end{aligned}$$

It follows from the inversion theorem that $P_u * P_v = P_{u+v}$. $\square$

Theorem 2.3. $||F(\cdot+iu)||_p$ increases as $u \downarrow 0$, for any $p, 1 \le p < \infty$ (if $p = 1$, consider $P_u * \mu$). Similarly, if f is bounded, $\sup_{x\in\mathbb{R}}|F(x+iu)|$ increases as $u \downarrow 0$.

Proof: Let $v < u$ be given. Let $r = u - v \ge 0$. Then

$$||P_u * f||_p = ||P_{v+r} * f||_p = ||(P_r * P_v) * f||_p \le ||P_r||_1||P_v * f||_p = ||P_v * f||_p \quad \square$$

Lemma 2.1. Let $f_u(x) = F(x+iu)$ be a harmonic function in the upper half plane such that

$$\sup_{u>0}||f_u(\cdot)||_p = A < \infty$$

Then

$$f_{u+v}(x) = (P_u * f_v)(x)$$

Proof: $f_{u+v}(x) = (P_u * f_v)(x)$ says that the values of $F(u+ix)$ at the level $u+v$ are the values of $F(u+ix)$ at the level v convolved with the Poisson kernel with parameter u.[7]

We may assume that F is real. Fix $v > 0$. Define $G(x+iu) = P_u * f_v(x)$ (G is the Poisson integral of the values of F at level v). $G(x+iu)$ is harmonic in $u > 0$ and $\sup_{u>0}||G(\cdot, u)||_p \le ||f_v(\cdot)||_p < \infty$. Note that $G(x+iu)$ has boundary value (pointwise limit) $f_v(x)$ as $u \to 0$, which can be simply viewed as the value of $G(x+iu)$ when $u = 0$. Therefore, $G(x+iu) - F(x+iu+iv)$ is a harmonic function in $u > 0$, satisfying $\sup_{u>0}||G(\cdot+iu) - F(\cdot+iu+iv)||_p < \infty$, continuous on the closed upper half plane and null on the real axis $u = 0$. Now, let

$$H(x+iu) = G(x+iu) - F(x+iu+iv)$$

We must show that $H(x+iu)$ vanishes for $u > 0$.

Let $\varphi \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$, where q is the conjugate exponent of p. Define

$$L(x+iu) = \int_{-\infty}^{\infty}\varphi(x-y)H(y+iu)dy$$

[6] If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$ and $g \in L^q(\mathbb{R})$, $\frac{1}{p}+\frac{1}{q} = 1$, then $(f * g)(x)$ exists everywhere, belongs to $C(\mathbb{R})$ and $||f * g||_c \le ||f||_p||g||_q$.

[7] In periodic case, $P_r * f_s = f_{rs}$ is proved by using the fact that a harmonic function is the real part of an analytic function.

Then $L(x + iu)$ is continuous on the closed upper half plane, harmonic in the upper half plane, Therefore, $\{f_u\}, u \quad 0,$ is bounded in $L^1(\mathbb{R})$

Su□ciency: By assumption, $||f_u||_1 \leq K,$ i.e., $||f_u(x)dx||_{M(\mathbb{R})} = ||f_u||_1 \leq K, \quad \forall u \quad 0$ Since $C_0(\mathbb{R}),$ as the pre-dual of $M(\mathbb{R}),$ is separable normed space, by Banach-Alaoglu theorem the closure of $\{f_u(x)dx\}$ in $M(\mathbb{R})$ is weak* sequentially compact. Therefore, there is a subsequence $\{f_{v_j}\}(x)dx$ of $f_u(x)dx$ that converges to some $\quad \in M(\mathbb{R})$ in weak* topology. That is,

$$\int (e^{-it}) f_{v_j}(t)dt \to \int (e^{-it})d \;(t), \qquad v_j \to 0$$

for each $\quad \in C_0(\mathbb{R})$ In particular, since for each $x, P_u(x-t) \in C_0(\mathbb{R}),$

$$\int P_u(x-t) f_{v_j}(t)dt \to \int P_u(x-t)d \;(t), \; v_j \to 0$$

On the other hand,

$$P_u * f_{v_j}(x) = f_{u+v_j}(x) \to f_u(x), \; v_j \to 0$$

Hence, $f_u(x) = \int P_u(x-t)d \;(t)$ for all x

We show that $|| \;|| = \lim_{u \downarrow 0} A_u$ Note that $\quad = \lim_{j\to\infty} f_{v_j}(x)dx$ in the weak* topology of $M(\mathbb{R})$ as the dual of $C_0(\mathbb{R})$ It follows that $|| \;|| \leq \liminf_{j\to\infty} A_{v_j}$ where $A_{v_j} = ||f_{v_j}||_1$ (For a proof, see the Appendix). Since A_u increases with $u \downarrow$ and $A_u \leq K, \; || \;|| \leq \lim_{u\to 0} A_u$ Furthermore, the inequality cannot be strict. Note that $f_u = P_u *$ and $||f_u||_1 \leq ||P_u||_1|| \;||$ Therefore, $A_u = ||f_u||_1 \leq || \;||$ for every $u \quad 0$ If the inequality were strict, we would have $A_u \leq || \;|| \quad \lim_{u\to 0} A_u$ for $u \quad 0,$ which is impossible.

As to the norm convergence of $||f_u - \;||_{M(\mathbb{R})} \to 0$ as $u \to 0,$ if $\quad$ is absolutely continuous then $\quad = f(x)dx$ for some $f \in L^1(\mathbb{R})$ Hence $f_u = P_u *$ is indeed $f_u = P_u * f$ Thus, by Fejer's theorem, $||f_u - f||_1 \to 0$ That is, $||f_u - \;||_{M(\mathbb{R})} \to 0$ as $u \to 0$ □

3. The Plancherel Theorem

In this section we define

$$\hat{f}(y) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} f(x)e^{-ixy}dx$$

Lemma 3.1. Let $\mathcal{C}$ be the collection of continuously di□erentiable functions with compact support. Then $\mathcal{C} \subset L^1(\mathbb{R}) \quad L^2(\mathbb{R})$ and $\mathcal{C}$ is a dense subspace of $L^2(\mathbb{R})$

Proof: Let $f \in L^2(\mathbb{R})$ Define $f_k(x) = f(x)$ if $|x| \leq \;$; and $f_k(x) = 0$ if $|x| \quad 0$ Then $f_k \to f$ in $L^2(\mathbb{R})$ Furthermore, we may choose an approximate identity with compact support and continuous derivative, for instance, let $\;(x) = e^{-\frac{1}{x^2}}$ for $x \geq 0$ and $\;(x) = 0$ for $x \quad 0$ Then $\quad \in C^\infty(\mathbb{R})$ and $\;(x) = \;(x+1)\;(1-x) \in C^\infty(\mathbb{R})$ and has compact support $[-1, 1],$ and $\;(x)dx = 1,$ when properly normalized. Let $e_n(x) = n\;(nx)$ Then $e_n(x)$ is an approximate

id[illegible]

Proof: Let $f \in \mathcal{C}$ Define $\tilde{f}(x) = \overline{f(-x)}$ Then $f * \tilde{f}(x) \in \mathcal{C}$ By the inversion theorem, at every point x where $(f * \tilde{f})(x)$ satisfies the Lipschitz condition, we have

$$(f * \tilde{f})(x) = \lim_{A,B\to\infty} \frac{1}{\sqrt{2}} \int_{-B}^{A} \widehat{f * \tilde{f}}(y) e^{ixy} \, dy$$

Since $f * \tilde{f}(x) \in \mathcal{C}$, it satisfies the Lipschitz condition at every point, in particular, at $x = 0$, we have

$$(f * \tilde{f})(0) = \lim_{A,B\to\infty} \frac{1}{\sqrt{}}$$

Corollary 3.1 (Inversion Theorem). If $f \in L^1(\mathbb{R})$ so that $\hat{f} \in L^1(\mathbb{R})$, then for a.e. x,

Lemma 3.8 (Multiplication Formula for $L^2(\mathbb{R})$). If $f, g \in L^2((\mathbb{R})$ then

$$\hat{f}g = \quad f\hat{g} \tag{3.4}$$

Proof: Fix $g \in L^1(\mathbb{R}) \quad L^2(\mathbb{R})$ first. Let $f \in L^2(\mathbb{R})$ and $f_k \in L^1(\mathbb{R}) \quad L^2(\mathbb{R})$ with $f_k = f$ Since $\hat{g} \in L^2(\mathbb{R}), \quad f_k\hat{g} \to \quad f\hat{g}$ It follows from the multiplication formula for $L^1(\mathbb{R})$ that $f_k\hat{g} = \quad \hat{f}_k g \to \quad \hat{f}g$ Hence for $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \quad L^2(\mathbb{R}), \quad \hat{f}g = \quad f\hat{g}$ Starting with this formula, for $f, g \in L^2(\mathbb{R})$, we approximate g by $g_k \in L^1(\mathbb{R}) \quad L^2(\mathbb{R})$ □

Theorem 3.2 (Plancherel). The Fourier transform $\mathcal{F}$ is a unitary operator of $L^2(\mathbb{R})$ and the inverse Fourier transform, $\mathcal{F}^{-1}$, can be obtained by $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$ for all $f \in L^2(\mathbb{R})$

Proof: We have already proved that $\mathcal{F}$ is an isometry, we only need to show $\mathcal{F}$ maps $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$, i.e., $E = \{\mathcal{F}(f) : f \in L^2(\mathbb{R})\} = L^2(\mathbb{R})$ As proven before, E is closed. Assume that $E \neq L^2(\mathbb{R})$ Then there exists $g \neq 0, g \in L^2(\mathbb{R}) \setminus E$, such that $g, f = 0$ for all $f \in E$, or $g, \hat{} = 0$ for all $\in L^2(\mathbb{R})$ It follows from the multiplication formula that $\overline{\hat{g}} = 0$ for all $\in L^2(\mathbb{R})$ In particular, taking $= \hat{g} \in L^2(\mathbb{R})$, $||\hat{g}||_2 = 0 = ||g||_2$ and $g = 0$ a.e., contrary to the assumption $g \neq 0$ Therefore, $\mathcal{F}$ is onto and so is a unitary operator of $L^2(\mathbb{R})$ □

4. Appendix

4.1. Weak/Weak * Topologies in Linear Spaces. Let X be a topological linear space and X' be its conjugate space of all continuous linear functionals on X [9]

The weak topology (X, X') on X is defined as follows:
Let F be a nonempty finite subset of X' Define

$$p_F(x) = \quad ax_{x' \in F} |x'(x)|, \quad x \in X$$

$p_F(x)$ is a seminorm on X (X, X') is the locally convex topology on X defined by the family of all seminorms $p_F(x)$, where F ranges over all finite subsets of X' A base at $x_0 \in X$ for this topology is given by sets of the form

$$x$$

A sequence $\{x_n\}$

Given a weak neighborhood = b,(0) of 0, can we always find $x_{mn} \in A$ so that $x_{mn} \in$? Observe that | x_{mn}, b | = $|b^{(m)}$ + $b^{(n)}|$, which can be made as small as we wish. First we choose large enough so that $|b^{(m)}|$ is very small, then for this fixed , choose n large enough so that | $b^{(n)}|$ is also very small.

Can we prove that there is no sequence of elements in A that converges weakly to 0? Given any sequence of elements in A, we show that there exist $_0$ 0 and $b \in {}^2$ (i.e. there exists a weak neighborhood b,$_0$(0) of 0) such that for any , we can always find an element a in this sequence with subscript $\geq$ such that $a \notin$ b,$_0$(0)

Consider a sequence, , of elements of $x_{mn} \in A$ If some integer, say , appears infinitely many times as the -index of $x_{mn} \in$, then we choose b so that $b^{(l)}$ = 1, $b^{(k)}$ = 0 $\neq$ Of course, $b \in {}^2$ and there is a (of course, infinite) subsequence $\{x_{ln}\}$ of with | b, x_{ln} | = 1 If none of the integers appears infinitely many times as -index in , then the range of -index of elements $x_{mn} \in$ is unbounded. We may extract a subsequence, call it , of so that their -indices form a (strictly) increasing sequence. Note that the range of n-index of $x_{mn} \in$